

Network Formation of Coalition Loyalty Programs

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1. INTRODUCTION

Loyalty programs are widely used in consumer retail, primarily serving as a mechanism for customer acquisition and retention [14, 15]. The most common forms of loyalty programs are *frequency reward programs*, such as airline frequent flyer programs, wherein customers earn certain number of *points* from every purchase. These points can be subsequently redeemed for rewards. The value of these rewards typically increases faster with the number of points – this non-linearity in rewards is what incentivizes members of a program to become more loyal to the merchant¹.

Over time, many stand-alone loyalty programs have agglomerated into larger coalition programs which allow customers to earn and redeem points across merchant partners in the coalition at specified exchange rates. The observed coalition networks are surprisingly complex (cf. Fig. 1), encompassing both pairwise partnerships as well as more centralized *coalition loyalty programs* such as Star Alliance and OneWorld (international airline alliances), Nectar (U.K.), Air Miles (Canada), Payback (Germany), Fly Buys (Australia), etc. Moreover, there is great variety in the exchange rates observed between different programs.

Though coalition loyalty programs are very popular and well studied in the literature [4, 3, 12], there is little formal understanding of the structure and strategic formation of such networks, specifically the *exchange rates between different programs*. Our work aims to address this gap.

Modeling Coalition Loyalty Programs: A network of loyalty programs can be viewed as a weighted directed graph, with nodes corresponding to merchants, and an edge from a merchant A to merchant B with weight r_{AB} corresponding to an agreement via which customers can convert 1 point issued by merchant B into r_{AB} points issued by merchant A . We henceforth refer to A as the source node and B as the sink node of this edge² and points issued by A as A -points and points issued by B as B -points.

We aim to understand the structure of these coalition programs. We do so via studying a strategic network formation game. Our model incorporates the following critical aspects of these networks:

¹For example, every major airline offers tiered rewards with a premium status on attaining sufficient miles.

²These exchange rates are public, and many of them can be collected from the service: <http://www.webflyer.com>.

- The non-linear nature of rewards which encourages customers to convert all points into their ‘home program’.
- Presence of an edge (A, B) increases demand for B ’s services by A ’s loyal customers, thereby increasing B ’s revenues.
- Conversely, A may incur a cost due to lost sales, in particular, if B is a competitor.
- A higher exchange rate r_{AB} leads to higher demand at B from A ’s customers, as it earns them a higher number of A -points. Moreover, if there are multiple (possibly multi-hop) paths between A and B , then A ’s loyal customers will use paths with the highest product of exchange rates to maximize the number of A -points they receive.
- Loyalty points are also a source of liability for the issuing merchant, as they can be redeemed in future by the customers [5]. Hence by permitting the formation of the edge (A, B) , A makes a commitment to accept a share of B ’s liability, for which B needs to compensate A .

Finally, a critical operational aspect of these networks is that they are often formed via negotiations between merchants, and the resulting contracts can not be easily modified. Moreover, in the absence of a central agency, these contracts are usually negotiated bilaterally between merchant pairs, which has a much lower setup cost than forming a centralized coalition.

Our Contributions: We study strategic network formation of coalition loyalty programs, under a model which incorporates all the above aspects. In Section 2 we present the model, and fully characterize the Nash Bargaining solution for two merchants. We also show that Nash Bargaining maximizes the social welfare.

In Section 3 we extend the model to multiple merchants, where we first show via a counterexample that bilateral Nash Bargaining no longer maximizes social welfare, thereby indicating that centralized coordination is useful in some settings. Nevertheless, we show that both social welfare maximization and bilateral Nash Bargaining lead to structurally identical networks: complete K -partite graphs, where the merchants within each partition are competitors, but across partitions they do not compete. Moreover, under mild conditions, we show that the sub-optimality in social welfare under bilateral Nash Bargaining is small. A particularly interesting case is where the merchants are completely heterogeneous and mutually non-competing. In this case it turns out that bilateral Nash Bargaining does maximize the social welfare, and a complete directed graph emerges as the solution.

Thus, in a nutshell, our results suggest that bilateral ne-

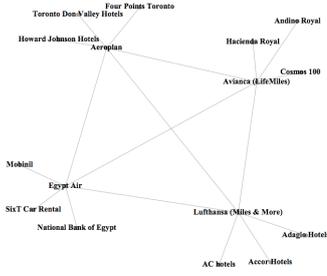


Figure 1: Extended Partners of Star Alliance Members: Shown are 4 Star Alliance members (Aeroplan, Avianca, Lufthansa, and Egypt Air) and their partnerships with local airline and hotel chains in addition to mutual partnerships between each other. An edge between two nodes indicate that the two merchants allow mutual exchange of reward points. The local partners of two different airlines usually do not form any mutual partnerships.

gotiation leads to similar network structures as centralized coordination, and achieves optimal or near-optimal welfare. We complement our results by collecting and displaying data on coalition loyalty programs, which validate our theoretical predictions.

Related Work: The management and impact of loyalty programs is well studied [3, 15] – however, the literature on coalition loyalty-programs is primarily empirical [9]. [12] provides a survey of the evolving structures of airline coalitions, suggesting a shift from bilateral ties to more connected structures over time.

Strategic network formation models have been used to study friendships in social networks [11], labor markets [2], etc. – see [10] for a good survey. Our work is closest in spirit to the models of directed network formation with unilateral decision making [1].

Our work is also tangentially related to the literature on credit networks [8]. In these networks, various authors have studied liquidity [6], strategic formation [7], and credit updating [13]. Loyalty-program networks have features which are quite distinct from credit networks, in particular, the notion of liability and the presence of exchange rates. To the best of our knowledge, our work is the first attempt at formalizing the strategic aspects of these networks.

2. BILATERAL NEGOTIATION MODEL

We start by considering the case of two merchants, A and B , who both run individual loyalty programs with their own loyal customer base, and are trying to negotiate a coalition loyalty program.

Let us first understand why A and B may both benefit from having a joint program. Suppose that A 's loyal customers (whom we henceforth refer to as type- A customers) occasionally want to avail services from B – arguably they are more likely to go to B for these services if the points that they earn from B (henceforth, B -points) upon purchase can be converted back to A -points. This excess demand is clearly beneficial to B – it not only brings in immediate revenues, but possibly future revenues from type- A customers preferring B over its competitors. Moreover, the likelihood of type- A customers bringing business to B will in general increase with the exchange rate r_{AB} , i.e., the number of A -points earned by exchanging 1 B -point. Hence, higher the exchange rate, the better it is for B in this respect.

On the other hand, by allowing B -point to A -point conver-

sions, A is taking on B 's liability of providing value against the points accumulated by a type- A customer using B . The higher the conversion rate, the higher is the volume of conversions, and thus higher is this liability. Further, every visit of A 's customer to B results in some lost sales (current/future), in particular, if B is a competitor. This can be viewed as a cost for A , which is high if A and B are competitors, and low if they offer complementary services. Moreover, it is natural for B to compensate A for taking on its liability. For similar reasons, A may also benefit if B allows conversions of A -points to B -points. The exact exchange rates and payments made for each point converted are decided via a negotiation between A and B .

Without loss of generality we can assume that the points in the two reward programs are normalized, so that both A and B bear a cost of p dollars for each point redeemed by a customer. Suppose that the exchange rates are r_{AB} and r_{BA} . One concern in any exchange network is the presence of arbitrage opportunities – to prevent this, we require the product of r_{AB} and r_{BA} to be not more than 1 (since otherwise customers can increase their points unboundedly for free). In fact, in most cases, the observed exchange rates between loyalty programs are usually less than 1. Hence we impose a stronger requirement that all rates lie in $[0, 1]$ ³.

Next, we assume that the rate at which points are converted from B to A , by type- A customers buying from B , is an affine function of r_{AB} : $\theta_{AB}r_{AB} + \gamma_{AB}$ (and similarly, we define $\theta_{BA}r_{BA} + \gamma_{BA}$ to be the rate of flow of points in the opposite direction). $\theta_{AB} + \gamma_{AB}$ can be thought of as the rate at which type- A customers visit B and exchange B -points for A -points under the highest possible exchange-rate (i.e., 1), while γ_{AB} is the rate at which type- A customers visit B when the exchange rate is 0. Note that this quantity allows us to abstract away the pricing by different merchants as well as their schemes to grant points to their customers. For simplicity we let γ values to be 0, although again our results remain the same if we relax this assumption.

Next, we define a_j be the value that merchant j ($j \in \{A, B\}$) obtains per point earned and converted by a type- i customer ($i \in \{B, A\}$). As mentioned earlier, this value can be viewed as a combination of immediate revenues due to the purchase, as well as future revenues due to the possibility that the customer may prefer j over its competitors in the future. On the other hand, we define c_{ij} to be the cost perceived by i for every point converted from j to i . Finally, we define q_{ij} to be the payment made by j to i for every point conversion initiated by a type- i customer from j to i .

To summarize: for $i \in \{A, B\}$, and $j \neq i$, we have:

- Rate of points-transfer from j to i : $\theta_{ij}r_{ij}$
- Payment received by i for the above transfer: $\theta_{ij}r_{ij}q_{ji}$
- Perceived cost of i for the above transfer: $\theta_{ij}r_{ij}c_{ij}$
- Cost of i for redeeming transferred liabilities: $\theta_{ij}r_{ij} \cdot r_{ij}p$
- Rate of points-transfer from i to j : $\theta_{ji}r_{ji}$
- Value received by i due to the above transfer: $\theta_{ji}r_{ji}a_i$
- Payment made by i for the above transfer: $\theta_{ji}r_{ji}q_{ij}$

Using the above, the utility derived by $i = A, B$ is:

$$u_i = -\theta_{ij}r_{ij}^2p - \theta_{ij}r_{ij}c_{ij} + \theta_{ij}r_{ij}q_{ji} + \theta_{ji}r_{ji}(a_i - q_{ij}). \quad (1)$$

And the social welfare, (i.e., the sum of utilities) is:

$$u_i + u_j = \theta_{ij}r_{ij}(a_j - c_{ij} - r_{ij}p) + \theta_{ji}r_{ji}(a_i - c_{ji} - r_{ji}p). \quad (2)$$

³Qualitatively our results do not change if we impose the weaker condition that product of pairwise exchange rates are bounded by 1

The quantity $a_j - c_{ij}$ is difference between the benefit that merchant j gets, and the perceived cost to merchant i , due to a type- i customer visiting j . This is the additional social welfare per transferred liability from j to i . We refer to this quantity as the unit potential value on edge (i, j) .

Solution Concept: We use *Nash Bargaining* as a solution concept to resolve bilateral negotiations between merchants. Under this, the directed exchange rates and the payments are chosen to maximize the product of the net utilities of the two merchants. The following result characterizes the Nash Bargaining solution in this setting (proof deferred to the appendix):

THEOREM 2.1. *Under the Nash Bargaining solution, we have for $i, j = A, B$,*

1. *If $a_j - c_{ij} \leq 0$, then $r_{ij} = 0$.*
2. *If $0 < a_j - c_{ij} \leq 2p$, then $r_{ij} = (a_j - c_{ij})/2p$.*
3. *If $a_j - c_{ij} > 2p$, then $r_{ij} = 1$.*

q_{ij} and q_{ji} are any values that satisfy:

$$\theta_{ij} r_{ij} (a_j - 2q_{ij} + c_{ij} + r_{ij} p) = \theta_{ji} r_{ji} (a_i - 2q_{ji} + c_{ji} + r_{ji} p).$$

Moreover, these parameters also maximize the social welfare.

Remarks: It is not in general true that the Nash bargaining solution maximizes social welfare, but it is in our case. To see why, note that the social welfare, which is the sum of the utilities of the two merchants, is independent of the payments q_{ij} . Further for any fixed value of social welfare, these payments can be designed in such a way such that the entire welfare becomes the utility of one merchant, while the other merchant gets 0 utility. Thus, in order to maximize the product of the utilities, it is necessary that the sum of the utilities is maximized.

The first condition means that the unit potential value on the edge from (i, j) is not positive, and thereby the exchange rate should be 0. On the other hand in the third case where the unit potential value is significantly larger than the servicing cost, the optimal exchange rate is the highest possible, i.e., 1.

It is easy to check that one possible set of payments is $q_{ij} = (a_j + c_{ij} + r_{ij} p_i)/2$ and $q_{ji} = (a_i + c_{ji} + r_{ji} p_j)/2$. However, there are other solutions as well. For instance if $c_{ij} = 0$ and $0 < a_j - c_{ij} \leq 2p$ for $i, j = A, B$, and further if $\theta_{ij} r_{ij} a_j = \theta_{ji} r_{ji} a_i$, then $q_{ij} = q_{ji} = 0$ is a solution as well. In this case, the reciprocal benefits exactly match for the two merchants and hence no payments are needed. In fact one can show that there is always a solution where one of q_{ij} or q_{ji} is 0, i.e., payments are unidirectional.

3. MODEL EXTENSIONS

We extend the model to more than two merchants trying to have a joint reward program. As we saw earlier, this involves deciding the directed exchange rates between every pair of merchants, and the payments made by the merchants to each other to compensate for the additional liabilities. But in this case a new feature of the problem arises. Suppose there are three merchants A, B , and C . If the direct exchange rate between two of them, say A and C , is lower than the ‘‘indirect’’ exchange rate obtained by first converting C -points into B -points and then converting those into A -points, i.e., $r_{AC} < r_{AB} \times r_{BC}$, then the direct exchange

rate r_{AC} is rendered defunct, since no customer will use it to convert points.

Not giving due consideration to this fact during the multi-party negotiation can have severe effect on the social welfare. For instance, this could happen if all the merchants set their exchange rates by resorting to pairwise bilateral negotiations, without considering the externality imposed by the decisions of others. Consider the example below.

Example: Let A and C be two competing airlines that operate between the same cities. We capture this by having $c_{AC} = a_C$, i.e., the value gained by C is exactly the value lost by A per point converted by a type- A customer from C to A . Let B be a hotel, which naturally does not compete with either A or C . We capture this by having $c_{AB} = c_{BA} = 0$ and $c_{CB} = c_{BC} = 0$. Further suppose that $a_A = a_B = a_C = a$. Since A and C are direct competitors, and $p > 0$, there is no additional cumulative welfare if some agents convert points between them. In this case, value is actually lost because of services provisioned against the points converted. Hence ideally, the exchange rates between them should be 0. This may not be possible because of the existence of the indirect exchange rate through B (since having a joint program with B may be welfare improving). But if A and B , and C and B undergo bilateral negotiations without considering this effect of their decisions, the resulting indirect exchange rate between A and C (through B) would typically be higher than the one that optimizes the welfare. We can show that in this case, the worst case ratio of the optimal social welfare and the welfare under bilateral Nash Bargaining is 1.58 if the maximum rate of points-transfer (θ) is the same between every merchant pair (Appendix 5.2). In fact, this worst case ratio holds true even for a large class of similar situations.

Let there be K classes of merchants, such that within each class the merchants are competitors, while across classes the merchants are non-competing. Assume that the value $a_j = a$ for all merchants. For any two merchants i and j within a class, suppose that $c_{ij} = c_{ji} = a$. For any two merchants i and j belonging to different classes, suppose that $c_{ij} = c_{ji} = 0$. And assume that the maximum rate of points-transfer (θ) is the same between every merchant pair. Then we have the following result:

PROPOSITION 3.1. *Under the above mentioned model, the worst case ratio of the optimal social welfare and the welfare under bilateral Nash Bargaining is 1.58.*

We conjecture that 1.58 is still an upper bound to the ratio of welfare values in the more general model where $a_j = a^\kappa$, where κ is the class of the merchant, and for any two merchants i and j within a class κ , we have $c_{ij} = c_{ji} = a^\kappa$. Moreover, in this more general model, we can show that the social welfare maximizing and pairwise bilateral Nash bargaining solutions lead to identical graphs structurally: the exchange rates are zero between every merchant pair within the same class, and they are non-zero across merchants belonging to different classes. We validate these results with our empirical observations (cf. Fig. 2).

Non-Competing Merchants

We now show that in the case where all merchants are non-competing, i.e., $c_{ij} = c_{ji} = 0$ for all i and j , the above issue does not arise, and somewhat surprisingly, pairwise bilateral Nash Bargaining leads to the social welfare maximizing outcome. The main point underlying this result is that if the

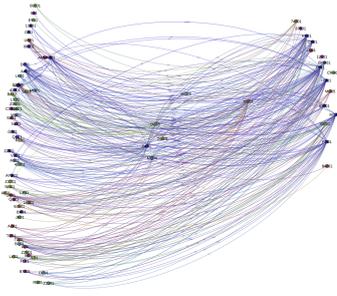


Figure 2: Partnerships Across Multiple Merchants: Nodes represent the merchants and an undirected edge between two nodes indicates that they mutually allow exchange of points. The graph shows three clear partitions: the one on the left are airlines, in the middle are credit card companies, and one on the right are hotel chains. Data collected by scraping <http://www.webflyer.com>.

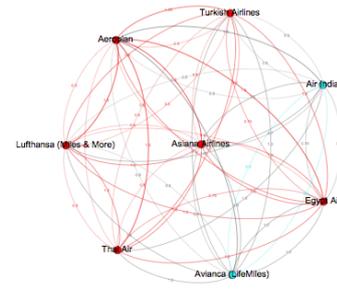


Figure 3: Exchange Rates within the Star Alliance Program: The graph is almost a complete graph (with the exception of Air India and Avianca Airlines) with different exchange rates between airline pairs. Direction of exchange rate is clockwise, and most of the rates depend only on the destination airline. Data collected by scraping different merchant websites.

exchange rates are set by bilateral Nash Bargaining when the merchants are non-competing, then for a customer, the direct exchange rate between any two merchants is at least as any indirect exchange rate.

CLAIM 1. *Let for any merchant pair, the exchange rates between them be set according to Theorem 2.1. Then for any three merchants i , j and k , $r_{ik} \geq r_{ij} \times r_{jk}$*

Observe that Theorem 2.1 implies that bilateral Nash Bargaining maximizes the sum of the utilities of all the merchants, assuming that only the direct exchange rates will be used by the customers (in this case the social welfare maximization problem decomposes across the different pairs of merchants). But if the customers are assumed to use the minimum exchange rate path between any two merchants, then the maximum sum of utilities of the merchants can only be lower. But the above claim implies that the solution obtained through bilateral Nash Bargaining naturally has the property that the direct exchange rates are higher than the indirect ones between every merchant pair. Summarizing, we thus have the following result:

THEOREM 3.1. *Suppose that merchants are non-competitive. Then exchange rates chosen by pairwise bilateral Nash Bargaining between different pairs of merchants maximize the social welfare.*

The above result indicates that pairwise negotiations are equivalent to centrally coordinated optimal solution, and this solution is a complete graph, where the exchange rate depends only on the destination merchant, when merchants are non-competing. This result is again consistent with empirical observations (cf. Fig. 3).

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5. APPENDIX

5.1 Proof of Theorem 2.1

PROOF. First observe the utility values of both merchants in this case:

$$u_i = -\theta_{ij}r_{ij}r_{ij}p - \theta_{ij}r_{ij}c_{ij} + \theta_{ij}r_{ij}q_{ji} + \theta_{ji}r_{ji}(a_i - q_{ij}) \quad (3)$$

$$u_j = -\theta_{ji}r_{ji}r_{ji}p - \theta_{ji}r_{ji}c_{ji} + \theta_{ji}r_{ji}q_{ij} + \theta_{ij}r_{ij}(a_j - q_{ji}) \quad (4)$$

Threat point is still $(0,0)$, and hence under Nash Bargaining $u_i u_j$ is maximized. This implies that the derivative of $u_i u_j$ w.r.t. q_{ij} and q_{ji} is 0 as these parameters are unconstrained. And for parameters being r_{ij} or r_{ji} , if the derivative of $u_i u_j$ is positive within the constraints of the exchange rates (i.e., $[0,1]$), then the value is maximized at the exchange rate equal to 1, otherwise, we find the exchange rate by setting the derivative to 0. Observe that for x being any of the parameters:

$$\frac{\partial u_i u_j}{\partial x} = u_j \frac{\partial u_i}{\partial x} + u_i \frac{\partial u_j}{\partial x} \quad (5)$$

Now we'll calculate $\frac{\partial u_i}{\partial x}$ and $\frac{\partial u_j}{\partial x}$ for each x :

$$\frac{\partial u_i}{\partial q_{ij}} = -\theta_{ji}r_{ji} \quad (6)$$

$$\frac{\partial u_j}{\partial q_{ij}} = \theta_{ji}r_{ji} \quad (7)$$

The above three equations immediately imply $u_i = u_j$. Now observe the following:

$$\frac{\partial u_i}{\partial r_{ij}} = \theta_{ij}(q_{ji} - c_{ij} - 2r_{ij}p) \quad (8)$$

$$\frac{\partial u_i}{\partial r_{ji}} = \theta_{ji}(a_i - q_{ij}) \quad (9)$$

$$\frac{\partial u_j}{\partial r_{ji}} = \theta_{ji}(q_{ij} - c_{ji} - 2r_{ji}p) \quad (10)$$

$$\frac{\partial u_j}{\partial r_{ij}} = \theta_{ij}(a_j - q_{ji}) \quad (11)$$

From the above equations we get the following:

$$\frac{\partial u_i u_j}{\partial r_{ij}} = u_i(\theta_{ij}(a_j - c_{ij} - 2r_{ij}p)) \quad (12)$$

Now we get the following cases:

1. If $a_j - c_{ij} \leq 0$: Eq. 12 is negative and hence the function value is maximized at $r_{ij} = 0$.
2. If $a_j - c_{ij} > 2p$: Eq. 12 is positive in the range of r_{ij} , and hence the function value is maximized at the maximum value of r_{ij} . That is $r_{ij} = 1$.
3. If $0 < a_j - c_{ij} \leq 2p$: Setting Eq. 12 to 0 we obtain $r_{ij} = \frac{a_j - c_{ij}}{2p}$.

The same result as above hold for r_{ji} . Also, equating u_i and u_j , we get the following:

$$\theta_{ij}r_{ij}(a_j - 2q_{ji} + c_{ij} + r_{ij}p) = \theta_{ji}r_{ji}(a_i - 2q_{ij} + c_{ji} + r_{ji}p) \quad (13)$$

It is easy to observe that one solution to the above equation is $q_{ji} = (a_j + c_{ij} + r_{ij}p)/2$ and $q_{ij} = (a_i + c_{ji} + r_{ji}p)/2$, substituting the appropriate values of r_{ij} and r_{ji} . \square

5.2 Analysis of Social Welfare Gap Example

Since the maximum rate of points-transfer is the same between every merchant pair, we can ignore it for calculating the ratio of social welfares. We also assume $a \leq 2p$. Let us analyze the optimal social welfare and the social welfare obtained via bilateral Nash Bargaining. First observe the social welfare (ignoring θ):

$$-r_{AB}^2 p + r_{AB} a - r_{BC}^2 p + r_{BC} a - (r_{AB} r_{BC})^2 p \quad (14)$$

Nash Bargaining Solution

Under bilateral Nash Bargaining, we get the exchange rates as $r_{AB} = r_{BC} = \frac{a}{2p} \leq 1$ and no edge from i to k (Theorem 2.1). For simplicity let $\frac{a}{2p} = t$. The social welfare in this case is:

$$\begin{aligned} & -\frac{a^2}{4p} + \frac{a^2}{2p} - \frac{a^2}{4p} + \frac{a^2}{2p} - \frac{a^4}{16p^3} \\ & = \frac{a^2}{2p} - \frac{a^4}{16p^3} \\ & = 2pt^2(1 - t^2/2) \end{aligned} \quad (15)$$

Optimal Welfare

We can argue that since there is symmetry, hence $r_{AB} = r_{BC}$ to maximize optimal social welfare. Let these two exchange rates be r . Then the social welfare is:

$$2(-r^2 p + ra) - r^4 p = p(-2r^2 + 4rt - r^4) \quad (16)$$

The above value is maximized at the value of r where the derivative of the function w.r.t. r is 0. The derivative is $p(-4r^3 - 4r + 4t)$. On substituting the root of this derivative as the value of r in the optimal welfare, and maximizing the ratio of optimal welfare and the Nash Bargain solution over t from 0 to 1, we find that the value is maximized at $t = 1$. The value of r obtained at $t = 1$ is around 0.68 and the maximized ratio value is around 1.58.

5.3 Proof of Proposition 3.1

First we prove the proposition for 2 classes of merchants A and B , and then extending to K classes is fairly straightforward. Let there be m merchants in A and n in B . The proof precedes very similar to the above simplified example.

Nash Bargaining Solution

Clearly the pairwise bilateral Nash Bargaining solution creates edges between merchant pairs across A and B with exchange rate $\hat{r} = \frac{a}{2p}$ (Theorem 2.1). This is assuming a is no more than $2p$. Also pairwise Nash Bargaining does not create an edge between merchant pairs within any partition as the unit potential value is negative. Let $\frac{a}{2p} = t \leq 1$. Consider three merchants A , B , and C , such that A and C are in the same partition and B is in the other partition, same as the above example. Overall social welfare is just the welfare obtained by these three merchants times the number of such merchant triplet combinations across the two partitions. The welfare obtained between these three merchants

can be calculated exactly like above. Thus the value is:

$$\begin{aligned} & (m(m-1)n + n(n-1)m) \times 2pt^2(1-t^2/2) \\ = & mn(m+n-2) \times 2pt^2(1-t^2/2) \end{aligned} \quad (17)$$

Optimal Welfare

Again because of symmetry, we can argue that exchange rates between any merchant pairs across the two partitions are the same. Let this quantity be r . And within any partition, having an exchange rate greater than r^2 will only hurt welfare. And having an exchange rate less than r^2 will never be used. Thus there are no edges between merchant pairs within any partition. Again like the preceding argument, the overall social welfare can be written as:

$$\begin{aligned} & (m(n-1)n + n(n-1)m) \times p(-2r^2 + 4rt - r^4) \\ = & mn(m+n-2) \times p(-2r^2 + 4rt - r^4) \end{aligned} \quad (18)$$

It is easy to see that the ratio of welfares is exactly as that in the preceding example, and hence the maximum value is the same 1.58.

5.4 Proof of Theorem 3.1

We first prove the Claim 1.

PROOF. The proof is fairly simple. We first show that for any three nodes i , j , and k , the following holds:

$$r_{ik} \geq r_{ij} \times r_{jk} \quad (19)$$

Observe that $r_{ik} = r_{jk} = \min\{\frac{a_k}{2p}, 1\}$ (Theorem 2.1). And $r_{ij} \leq 1$ by definition. Hence the above equation always holds. \square

Hence the exchange rate along any directed edge is the maximum among all paths between those two merchants. Thus all transactions happen via direct edges, and no transactions happen along hop lengths of more than 1. Now the proof of the theorem easily follows via the argumentation we provided just prior to writing it.