Competitive Rumor Spread in Social Networks*

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ABSTRACT

We consider a setting in which two firms compete to spread rumors in a social network. Firms seed their rumors simultaneously and rumors propagate according to the linear threshold model. Consumers have (randomly drawn) heterogeneous thresholds for each product. Using the concept of *cascade centrality* introduced by [6], we provide a sharp characterization of networks in which games admit pure-strategy Nash equilibria (PSNE). We provide tight bounds for the efficiency of these equilibria and for the inequality in firms' equilibrium payoffs. When the network is a tree, the model is particularly tractable.

Keywords

cascade centrality, competitive diffusion, rumor spread, threshold models

1. INTRODUCTION

Rumors often spread quickly through social networks and firms often compete to spread different rumors. In our model an agent's decision to a spread a rumor is made according to the linear threshold process [4] but the choice of rumor depends only on *latest* period spreaders (as in the *linear switching function* of [1]). A good of example of this process is Twitter which displayed tweets to chronological order. When each firm (or political party) has a unit budget i.e. can only seed to exactly one initial spreader, there may be no pure-strategy Nash equilibrium (PSNE) in general social networks. Informally, a PSNE may fail to exist if the social network is very symmetric and has many cycles. However, using *cascade centrality* [6], we are able to characterize PSNEs whenever they exist. Cascade centrality of an agent in a network measures the expected number of consumers who spread that rumor when the agent alone is the seed and there are no other competing rumors. In sufficiently asymmetric networks, there are certain agents – those with the highest cascade centrality – who are obvious candidates for seeding in a pure-strategy Nash equilibrium. Using cascade centrality, we give an explicit characterization of PSNEs. In general, PSNEs are inefficient: a social planner, who wants to maximize the total number of spreaders of either rumor can outperform competing firms. We show that the price of anarchy [5] – the ratio of the socially optimal number of spreaders to the number of spreaders in the worst PSNE – is at most 1.5. The motivation for studying social optimum is this case might be that having no information about an important issue, such as an upcoming referendum, is worse from a social point of view than spreading the view of one side and therefore being engaged in the political process. The highest ratio of equilibrium payoffs, known as the budget multiplier [2], is at most 2. Moreover, we illustrate that both of these bounds are tight (improving the PoA bound of [3]). To illustrate the tractability of our framework, we also consider the diffusion game in trees. We show that PSNEs always exist in trees and provide a complete characterization of their structure and efficiency properties.

*Full paper available at:
`http://t8el.com/wp-content/uploads/2015/06/Competition.pdf`

NetEcon 2016 Juan-les-Pins, France

2. MODEL

Our set up follows [3, 1]. $G(V, E)$ is a simple, undirected graph in which nodes V represent agents and E links between them. Each agent in the graph is endowed with a threshold drawn from $\mathcal{U}(0, 1)$. There two firms, A and B, which spread rumors a and b. An agent can be in one of three states: not spreading a rumor, spreading a or spreading b. Firms seed agent simultaneously and rumors spread as in the linear switching function model of [1]. At time $t = 0$, no agents are spreading either rumor and each firm simultaneously chooses one agent each as a seed for its rumor. If firms seed the same agents, it picks one of the rumors to spread at random. In subsequent steps, each agent checks if in the previous period the proportion of his neighbours who are spreading *either* rumor is greater or equal to his threshold. If not, he remains as a non-spreader. If his threshold is exceeded, then the agent decides to spread one of the rumors with the probability that is equal to the fraction of his neighbors who *started spreading that rumor in the previous period.* Eventually, the rumor spreading cascades must stop and we can check all agents' states. The firms do not know the particular draw of agents' thresholds ex ante.

DEFINITION 1. *The cascade centrality of node $i \in V$ in graph G is defined as:*

$$\mathcal{C}_i(G) = 1 + \sum_{j \in V \setminus \{i\}} \sum_{P \in \mathcal{P}_{ij}} \frac{1}{\chi_P}$$

where χ_P is the product of the degrees of nodes (except i) along some simple path P that begins at i and end at j and $\mathcal{P}_{ij}$ is the set of all simple paths that begin at i and end at

j.[1]

Since the reference to G is usually unambiguous, we refer to $\mathcal{C}_i(G)$ as $\mathcal{C}_i$ and $d_i(G)$ as d_i. For $i \neq j$, let us denote $\Xi(i,j)$ as the set of all paths that begin at i and include (but do not necessarily end at) j.

$$\epsilon(i,j) = \sum_{P \in \Xi(i,j)} \frac{1}{\chi_P}$$

$\epsilon(i,j)$ describe the extent to which node j interferes with the cascade emanating from node i. Note that $\epsilon(i,j) \neq \epsilon(j,i)$ unless $d_j = d_i$, so a node with a higher degree interferes more with a node of a lower degree than vice versa.

2.1 Game and equilibrium

The action space of firms A and B is hence denoted $\Sigma := \Sigma_A \times \Sigma_B := V \times V$; action profile $\sigma := (\sigma_A, \sigma_B)$. Action profiles are therefore simply (ordered) pairs of nodes (agents) in the graph.

A firm's payoff is simply the expected number of spreaders of its rumor given the action of the other firm. For a payoff profile $\pi := (\pi_A(\sigma), \pi_B(\sigma))$, we can define the game $\Gamma := (\Sigma, \pi)$.

Define $\mathcal{P}^*(i,j) \subseteq \mathcal{P}(i,j)$ as the set of all paths from i to j that include at most one of the seeds along the path.

PROPOSITION 1. *Consider a duopoly with unit budgets Γ. The expected number of spreaders of rumor a (i.e. firm A's payoff) is*

$$\pi_A(\sigma_A, \sigma_B) := \sum_{j \in V} \sum_{P \in \mathcal{P}^*(\sigma_A, j)} \frac{\iota}{\chi_P}$$

where

$$\iota = \begin{cases} 1/2 & \text{if } \sigma_A = \sigma_B \\ 1 & \text{otherwise} \end{cases}$$

Alternatively, we can express firm A's payoff as

$$\pi_A(\sigma_A, \sigma_B) = \begin{cases} \mathcal{C}_{\sigma_A}/2 & \text{if } \sigma_A = \sigma_B \\ \mathcal{C}_{\sigma_A} - \epsilon(\sigma_A, \sigma_B) & \text{otherwise} \end{cases}$$

We want to focus on the pure-strategy Nash equilibria of this game.[2]

DEFINITION 2. *A profile of actions $\sigma^* := (\sigma_A^*, \sigma_B^*) \in \Sigma$ is a pure-strategy Nash equilibrium of Γ if:*

- $\pi_A(\sigma_A^*, \sigma_B^*) \geq \pi_A(\sigma_A, \sigma_B^*)$ *for all actions* $\sigma_A \in \Sigma_A$
- $\pi_B(\sigma_A^*, \sigma_B^*) \geq \pi_B(\sigma_A^*, \sigma_B)$ *for all actions* $\sigma_B \in \Sigma_B$

Define Σ^* as the set of all pure-strategy Nash equilibria of the game.

3. RESULTS

In general, a game Γ does not admit a PSNE. Figure 1 gives an example of such a network and its cycle best responses.

[1]We refer to simple paths as "paths" throughout.

[2]Since the number of firms and the strategy spaces are finite, a mixed-strategy Nash equilibrium always exists.

3.1 Existence and characterization of PSNE

We now characterize both types of PSNE: in which firms seed the same agent and in which firms seed different agents.

THEOREM 1. *Consider a duopoly with unit budgets Γ. Then Γ admits at least one PSNE if and only if at least one of the following conditions on G is satisfied:*

1. *There exists $i \in V$ such that, for any $j \in V \setminus \{i\}$:*
   - $\dfrac{\mathcal{C}_i}{\mathcal{C}_j} \geq 2 - 2 \cdot \left(\dfrac{\epsilon(j,i)}{\mathcal{C}_j}\right)$
2. *There exist $i, j \in V$ such that, $\mathcal{C}_i \geq \mathcal{C}_j$ and for any $k \in V \setminus \{i,j\}$:*
   - $\dfrac{\mathcal{C}_i}{\mathcal{C}_k} \geq 1 + \dfrac{\epsilon(i,j) - \epsilon(k,j)}{\mathcal{C}_k}$
   - $\dfrac{\mathcal{C}_j}{\mathcal{C}_k} \geq 1 + \dfrac{\epsilon(j,i) - \epsilon(k,i)}{\mathcal{C}_k}$
   - $\dfrac{1}{2} + \dfrac{\epsilon(i,j)}{\mathcal{C}_j} \leq \dfrac{\mathcal{C}_i}{\mathcal{C}_j} \leq 2 - 2 \cdot \left(\dfrac{\epsilon(j,i)}{\mathcal{C}_j}\right)$

If Condition 1 is satisfied then there exists a $\sigma^* = (i,i)$ PSNE and if Condition 2 is satisfied, then there exists a $\sigma^* = (i,j)$ (and $\sigma^* = (j,i)$ by symmetry) PSNE.

Although the conditions set out in Theorem 1 appear involved, they are rather intuitive.[3] Let us first focus on Condition 1. When firms seed the same agent i, their payoff is 0 with probability $\frac{1}{2}$ and $\mathcal{C}_i$ with probability $\frac{1}{2}$. To ensure that it is worthwhile for both firms to take that gamble, the cascade centrality of the node has to be sufficiently high. Indeed, it is sufficient that the cascade centrality of the agent with the second highest cascade centrality is at most half of $\mathcal{C}_i$ as the following corollary summarizes.

COROLLARY 1. *$\sigma^* = (i,i)$ is a PSNE of Γ if $\dfrac{\mathcal{C}_i}{\mathcal{C}_j} \geq 2$ for any $j \neq i$.*

Condition 2 is necessary and sufficient for the existence of a $\sigma^* = (i,j)$ PSNE with two different seeds. The first two parts of the condition say that the cascade centralities of i and j need to sufficiently high compared to other agents in the network. The final condition ensures that while the cascade centrality of i is sufficiently greater than j's, it is not so high that it incentivizes agents to move into the $\sigma^* = (i,i)$ equilibrium. In fact, we can summarize this intuition with a clean sufficient condition.

COROLLARY 2. *$\sigma^* = (i,j)$ (and $\sigma^* = (j,i)$ by symmetry) is a PSNE of Γ if for any $k \neq i,j$*

- $\mathcal{C}_i \geq \max\left\{\dfrac{\mathcal{C}_j}{2}, \mathcal{C}_k\right\} + \epsilon(i,j)$
- $\mathcal{C}_j \geq \max\left\{\dfrac{\mathcal{C}_i}{2}, \mathcal{C}_k\right\} + \epsilon(j,i)$

[3]Note that there is a knife-edge case in which there are both types of equilibria.

Note the role that ϵ – the degree of interference between the agents – plays. ϵ is small when the agents are "far" from each other - in terms of the lengths and numbers of paths and the degrees of agents along these paths. If interference is small, then a sufficient condition for a PSNE with different seeds is that the cascade centrality of the agent with the highest cascade centrality in the network is no more than twice the cascade centrality of the second-highest agent.

Finally, Theorem 1 sheds light on why there is no PSNE in the social network in Figure 1. Condition 1 clearly cannot be satisfied: there was no agent that has a far higher centrality than others. But Condition 2 could not be satisfied because there was a great deal of interference between the agents. There are several paths between i and j and $\mathcal{C}_k$ is very close to $\mathcal{C}_i$ and $\mathcal{C}_j$ making it difficult to satisfy any of the three parts of Condition 2.

3.2 Efficiency of equilibria: price of anarchy

We now to turn to the efficiency properties of the PSNEs described above. It is easy to find an example of a social network in which none of the PSNEs is efficient (see Figure 4). We will focus on the price of anarchy, introduced by [5], which measures the ratio of the number of spreaders in the socially optimal outcome to the lowest expected number of spreaders among all Nash equilibria. The social planner's objective is to maximize the sum of firms' payoffs. Let $Y(\sigma) := \pi_A(\sigma) + \pi_B(\sigma)$ be this objective.

Definition 3. *Price of Anarchy of* Γ *is defined as:*

$$PoA(\Gamma) = \frac{\max_{\sigma\in\Sigma} Y(\sigma)}{\min_{\sigma\in\Sigma^*} Y(\sigma)}$$

The following theorem puts a bound on how "bad" equilibrium outcomes can be.

Theorem 2. *Consider a duopoly with unit budgets* Γ. *For any* Γ *that admits at least one PSNE,*

$$1 \leq PoA(\Gamma) < 1.5.$$

This (upper) bound is also tight, which is shown in the lower network in Figure 3.

3.3 Inequality of payoffs: budget multiplier

We consider the notion of the budget multiplier introduced by [2]. This metric measures the extent to which the network amplifies an asymmetry in budgets into an asymmetry in payoffs. Even though the budgets are equal in our model, equilibrium payoffs are not equal in general.

Definition 4. *For arbitrary integer budgets* $\mathcal{B}_A$ *and* $\mathcal{B}_B$, *the budget multiplier of game* Γ *is defined as:*

$$BM(\Gamma) = \max_{\sigma\in\Sigma^*} \frac{\pi_A(\sigma)/\pi_B(\sigma)}{\mathcal{B}_A/\mathcal{B}_B}$$

So far, we have only analyzed the case of unit budgets, where $\mathcal{B}_A = \mathcal{B}_B = 1$. Hence, the budget multiplier in our case is simply $\max_{\sigma\in\Sigma^*} \frac{\pi_A(\sigma)}{\pi_B(\sigma)}$.

Theorem 3. *For any* Γ *that admits at least one PSNE,*

$$1 \leq BM(\Gamma) < 2$$

This (upper) bound is also tight as shown in the upper network in Figure 2.

3.4 Competition on trees

If G is a tree, we can write $\mathcal{P}(i,j) = P(i,j)$. Denote by $\Delta_1(G)$ ($\Delta_2(G)$) the degree of the agent with the (weakly second-) highest degree in the network. Cascade centrality of any node in a tree is its degree plus 1 [6].

Definition 5. *Candidate sets of strategy profiles are defined as:*

- $\sigma_0 = \{(i,i) | d_i = \Delta_1\}$.
- $\sigma_1 = \{(i,j) | (d_i, d_j) \in \{(\Delta_1, \Delta_2), (\Delta_2, \Delta_1)\}\}$.
- $\sigma_2 = \{(i,j) | (d_i, d_j) \in \{(\Delta_1, \Delta_2 - 1), (\Delta_1 - 1, \Delta_2), (\Delta_2, \Delta_1 - 1), (\Delta_2 - 1, \Delta_1)\}\}$.

For each set of strategy profiles indexed by $l \in \{1,2\}$ we can define the strategy profile that maximizes the degree sequence product among all strategy profiles in σ_l as $\sigma_l^* = \arg\max_{(i,j)\in\sigma_l} \chi_{P(i,j)}$. The value of this maximum degree sequence is denoted $\delta_l = \max_{(i,j)\in\sigma_l} \chi_{P(i,j)}$.

Proposition 2. *Suppose G is a tree and consider a duopoly with unit budgets* Γ. *Then,* Γ *admits a PSNE* σ^*, *which is characterized as follows:*

- *If* $\delta_1 = 1$, $\sigma^* = \begin{cases} \sigma_1^* & \text{if } \Delta_1 < 2\Delta_2 - 1 \\ \sigma_0 & \text{else if } \Delta_1 > 2\Delta_2 - 1 \\ \sigma_0 \cup \sigma_1^* & \text{otherwise} \end{cases}$
- *Else if* $\delta_1 = 2$, $\sigma^* = \begin{cases} \sigma_1^* & \text{if } \Delta_1/\Delta_2 < 2 \\ \sigma_0 & \text{else if } \Delta_1/\Delta_2 > 2 \\ \sigma_0 \cup \sigma_1^* & \text{otherwise} \end{cases}$
- *Otherwise,* $\sigma^* = \begin{cases} \sigma_1^* & \text{if } \Delta_1/\Delta_2 \leq 2 \\ \sigma_0 & \text{otherwise} \end{cases}$

In our case, we can also solve the social planner's problem on trees explicitly. Denote a solution by $\sigma^Y \in \arg\max_{\sigma\in\Sigma}\{Y(\sigma)\}$.

Proposition 3. *Suppose G is a tree. Then, if G is a star,* $\sigma^Y = \sigma_0 \cup \sigma_1^*$. *Otherwise,*

$$\sigma^Y = \begin{cases} \sigma_2^* & \text{if } \delta_1 = 1, \delta_2 > 2, \sigma_2^* \neq \emptyset \\ \sigma_1^* & \text{otherwise} \end{cases}$$

Corollary 3. *Suppose G is a tree. Then, there*

- *are no efficient PSNEs iff* $(\sigma^* = \sigma_0) \vee (\sigma^Y = \sigma_2^*)$
- *are no inefficient PSNEs iff* $(\sigma^* = \sigma_1^*) \wedge (\sigma^Y = \sigma_1^*)$
- *is an inefficient PSNE iff* $[(\sigma^* \supset \sigma_0) \wedge (\sigma^Y = \sigma_1^*)] \vee (\sigma^W = \sigma_2^*)$
- *is an efficient PSNE iff* $(\sigma^* \supset \sigma_1^*) \wedge (\sigma^Y = \sigma_1^*)$
- *is an efficient and an inefficient PSNE iff* $(\sigma^* = \sigma_0 \cup \sigma_1^*) \wedge (\sigma^Y = \sigma_1^*)$

Figure 4 shows an example of a network with both types of PSNEs: (i,i), $(i.k)$ and (j,i) (as well as symmetric counterparts of the latter two) because the largest degree (viz. 3) is exactly equal to twice the second-largest (viz. 2) minus one. None of the PSNEs is efficient.

4. REFERENCES

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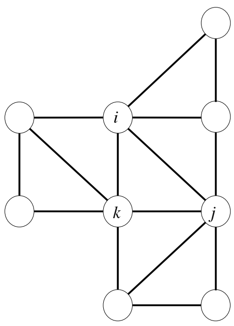
APPENDIX

A. FIGURES

In the graph on Figure 1, the agent with the highest cascade centrality is i. The pure-strategy best response to a firm's seeding i is to seed k. But the best response to k is j and, in turn, the best response to j is to seed i. Therefore, the only equilibria in this game are mixed-strategy.

Figure 1: Example of a social network for a game that does not admit a PSNE



In both graphs on Figure 2, (i, j) is a unique PSNE (of course, (j, i) is also a PSNE by symmetry). Suppose that we create a sequence of graphs, such that at every step of the sequence we increase the degree of i and j by d_i and d_j respectively. This means that along the sequence the ratio d_i/d_j remains constant at 2 and the equilibrium remains unique (up to a symmetrical transformation). However, the limit payoff of the firm seeding at i is d_i, which is twice the limit payoff d_j of the firm seeding at j, as the degrees of these nodes increase along the sequence.

In both graphs on Figure 3, the unique PSNE is one in which both firms seed agent i. Because the cascade centrality of i is so high, both firms are happy to take the gamble that involves getting a zero payoff. However, the social planner would prefer to seed i and j in both cases. It is easy to show that the expected number of spreaders in equilibrium is 5, whereas the social optimum approaches 7.5 as the length of the branch goes to infinity.

Figure 2: Social network sequence that reaches the highest budget multiplier.

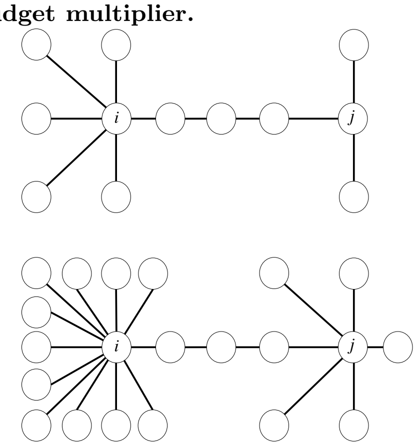


Figure 3: Social network sequence that reaches the highest price of anarchy.

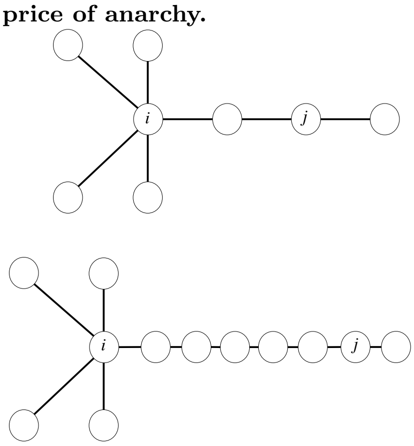
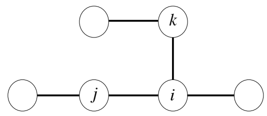


Figure 4: A tree in which none of the PSNEs is efficient



B. PROOFS

PROOF OF PROPOSITION 1. • When $\sigma_A = \sigma_B$, since the tie is broken with an equal probability for each firm, the problem is identical a single-seed case with a seeding probability of a half (see [?] for details). Hence, the payoff π_A takes the prescribed form.

- Suppose there are two seeds i, j. The probability that node $k \in V \setminus \{i, j\}$ spreads the rumor is

$$\sum_{P \in \mathcal{P}^\star(i,k)} \frac{1}{\chi_P} + \sum_{P \in \mathcal{P}^\star(j,k)} \frac{1}{\chi_P}$$

The probability that k is influenced by a live-edge path (see [4] for definition) from seed i (of firm A) is $\sum_{P \in \mathcal{P}^\star(i,k)} \frac{1}{\chi_P}$. But $\mathcal{P}(i,k) = \mathcal{P}(i\text{–via } j\text{–}k) \cup \mathcal{P}^\star(i,k)$ (where, $\mathcal{P}^\star(i,k)$ is defined as all paths from i to k not via the other seed j as in the main text), so probability that k is influenced by i can be written as

$$\sum_{P \in \mathcal{P}(i\text{–via } j\text{–}k)} \frac{1}{\chi_P} + \sum_{P \in \mathcal{P}^\star(i,k)} \frac{1}{\chi_P}$$

Summing over all $k \in V$, gives us

$$\mathcal{C}_i = 1 + \sum_{k \in V} \sum_{P \in \mathcal{P}(i\text{–via } j\text{–}k)} \frac{1}{\chi_P} + \sum_{k \in V} \sum_{P \in \mathcal{P}^\star(i,k)} \frac{1}{\chi_P}$$

Therefore,

$$1 + \sum_{k \in V} \sum_{P \in \mathcal{P}^\star(i,k)} \frac{1}{\chi_P} = \mathcal{C}_i - \sum_{k \in V} \sum_{P \in \mathcal{P}(i\text{–via } j\text{–}k)} \frac{1}{\chi_P}$$

where $1 + \sum_{k \in V} \sum_{P \in \mathcal{P}^\star(i,k)} \frac{1}{\chi_P} \equiv \pi_A(i,j)$ and $\sum_{k \in V} \sum_{P \in \mathcal{P}(i\text{–via } j\text{–}k)} \frac{1}{\chi_P} \equiv \epsilon(i,j)$ giving us the desired expression for the expected number of agents affected by seed i of firm A. The expression for seed j (of firm B) is analogous.

□

PROOF OF THEOREM 1. Fix a duopoly with unit budgets Γ. Then, there are two types of PSNE:

- Type 1: (i,i) for some $i \in V$.
- Type 2: (i,j) for some $i \neq j \in V$.

By Definition 2, for some $i \in V$, (i,i) is a type 1 PSNE if and only if

- $\pi_A(i,i) \geq \pi_A(j,i)$ for all $j \in V$,
- $\pi_B(i,i) \geq \pi_B(i,j)$ for all $j \in V$.

By Proposition 1, these conditions hold if and only if $\frac{\mathcal{C}_i}{2} \geq \mathcal{C}_j - \epsilon(j,i)$, which is precisely the condition 1 upon re-writing.

Now, for some $i \neq j \in V$, (i,j) is a type 2 PSNE if and only if

- $\pi_A(i,j) \geq \pi_A(k,j)$ for all $k \in V$,
- $\pi_B(i,j) \geq \pi_B(i,k)$ for all $k \in V$.

By Proposition 1, these conditions hold if and only if

- $\mathcal{C}_i - \epsilon(i,j) \geq \mathcal{C}_k - \epsilon(k,j)$ for all $k \neq i, j$ and $\mathcal{C}_i - \epsilon(i,j) \geq \mathcal{C}_j/2$,
- $\mathcal{C}_j - \epsilon(j,i) \geq \mathcal{C}_k - \epsilon(k,i)$ for all $k \neq i, j$ and $\mathcal{C}_j - \epsilon(j,i) \geq \mathcal{C}_i/2$,

which is precisely the condition 2 upon re-writing. □

PROOF OF COROLLARY 1. We apply a condition 1 from Theorem 1. Suppose $\mathcal{C}_i/\mathcal{C}_j \geq 2$ for any $j \neq i$. Then, $\mathcal{C}_i/\mathcal{C}_j \geq 2 \geq 2 - 2 \cdot \epsilon(j,i)/\mathcal{C}_j$ since $0 \leq \epsilon(j,i) \leq \mathcal{C}_j$. □

PROOF OF COROLLARY 2. We apply a condition 2 from Theorem 1.

Suppose $\mathcal{C}_i \geq \max\{\mathcal{C}_j/2, \mathcal{C}_k\} + \epsilon(i,j)$. Then, $\mathcal{C}_i \geq \mathcal{C}_k + \epsilon(i,j) \geq \mathcal{C}_k + \epsilon(i,j) - \epsilon(k,j)$. This is the first condition upon dividing by $\mathcal{C}_k$. The second condition can be treated similarly.

Also, $\mathcal{C}_i \geq \max\{\mathcal{C}_j/2, \mathcal{C}_k\} + \epsilon(i,j)$ forces $\mathcal{C}_i \geq \mathcal{C}_j/2 + \epsilon(i,j)$, which is the first half of the third condition. The second half is analogous. □

PROOF OF THEOREM 2. LEMMA 1. *Suppose (i,j) is a PSNE where $i \neq j \in V$. Then,*

$$Y(i,j) \geq \frac{\mathcal{C}_i + \mathcal{C}_j}{2}.$$

PROOF OF LEMMA 1. Since (i,j) is a PSNE, in particular,

$$\pi_A(i,j) \geq \pi_A(j,j) = \frac{1}{2} \cdot \mathcal{C}_j \text{ and}$$
$$\pi_B(i,j) \geq \pi_B(i,i) = \pi_A(i,i) = \frac{1}{2} \cdot \mathcal{C}_i.$$

Hence, $Y(i,j) = \pi_A(i,j) + \pi_B(i,j) \geq \frac{1}{2} \cdot (\mathcal{C}_i + \mathcal{C}_j)$. □

LEMMA 2. *For any $j \neq i$,*

$$\epsilon(j,i) \leq \frac{1}{2}\mathcal{C}_j.$$

PROOF OF LEMMA 2. Let $\delta(j,i) = \sum_{p \in \Lambda(j,i)} \frac{1}{\chi_p}$ where $\Lambda(j,i)$ is a set of all path starting from j that excludes i. Then, $\mathcal{C}_j = \epsilon(j,i) + \delta(j,i)$. So, the statement amounts to showing $\epsilon(j,i) \leq \delta(j,i)$.

Now, for each path $p \in \Xi(j,i)$, we may decompose it as $q - i - r$ where $q = (q_0, \cdots, q_u) \in \Lambda(j,i)$ is a path with $q_j \neq i$ for all $j \in \{0, \cdots, u\}$ and $r = (r_0, \cdots, r_v)$ is a path with $r_0 \in N_i$. Therefore, the statement follows from the following estimate:

$$\begin{aligned}
\epsilon(j,i) &= \sum_{p \in \Xi(j,i)} \frac{1}{\chi_p} = \sum_{p = q - i - r \in \Xi(j,i)} \frac{1}{\chi_p} \\
&\leq \sum_{q \in \Lambda(j,i)} \frac{1}{\chi_q} \cdot \frac{1}{d_i} \cdot \left(\sum_{r \text{ where } r_0 \in N_i - \{q_u\}} \frac{1}{\chi_r} \right) \\
&\leq \sum_{p \in \Lambda(j,i)} \frac{1}{\chi_p} \cdot \frac{\mathcal{C}_i - 1}{d_i} \\
&\leq \sum_{p \in \Lambda(j,i)} \frac{1}{\chi_p} \cdot \frac{(d_i + 1) - 1}{d_i} = \sum_{p \in \Lambda(j,i)} \frac{1}{\chi_p} = \delta(j,i).
\end{aligned}$$

□

COROLLARY 4. *For any $i, j \in V$, $\pi_A(i,j) \geq \mathcal{C}_i/2$.*

PROOF OF COROLLARY 4. This follows from Lemma 2 coupled with the formula for the payoff. □

Lemma 3. *Suppose (i,j) is a social optimum and i, j, and k are pairwise distinct. Then,*

$$\epsilon(i,k) - \epsilon(i,j) + \epsilon(j,k) - \epsilon(j,i) < \frac{1}{2}\mathcal{C}_k.$$

Proof of Lemma 3. For a fixed n, the graph configuration that minimizes $\frac{1}{2}\mathcal{C}_k - (\epsilon(i,k) - \epsilon(i,j) + \epsilon(j,k) - \epsilon(j,i))$ is a path of length n. It is easy to check for a path. $\square$

Let σ^* be a PSNE, σ^Y be a social optimum, and ρ be the ratio $Y(\sigma^Y)/Y(\sigma^*)$. For any choice of σ^* and σ^Y, it suffices to show $\rho < 1.5$.

Suppose first that $\sigma^* = (i,i)$. There are few cases to consider depending on σ^Y. Indices i, j, k, and l are assumed to be pairwise distinct.

1. $\sigma^Y = (i,i)$. Then, $\rho = 1 < 1.5$.
2. $\sigma^Y = (i,j)$. Then, since (i,i) is a PSNE,
$$\pi_A(i,j) < \mathcal{C}_i \text{ and } \mathcal{C}_i/2 = \pi_B(i,i) \geq \pi_B(i,j).$$
Hence, $\rho < (\mathcal{C}_i + \mathcal{C}_i/2)/\mathcal{C}_i = 1.5$.
3. $\sigma^Y = (j,j)$. Then, since (i,i) is a PSNE, by Lemma 2,
$$\mathcal{C}_i/2 = \pi_A(i,i) \geq \pi_A(j,i) = \mathcal{C}_j - \epsilon(j,i) \geq \mathcal{C}_j/2.$$
Hence,
$$\rho = \frac{\mathcal{C}_j}{\mathcal{C}_i} \leq 1 < 1.5.$$
4. $\sigma^Y = (j,k)$. Then, since (i,i) is a PSNE, for any l, $\pi_A(l,i) \leq \pi_A(i,i) = \mathcal{C}_i/2$. So, $\mathcal{C}_l \leq \mathcal{C}_i/2 + \epsilon(l,i)$. Hence,
$$\pi_A(j,k) = \mathcal{C}_j - \epsilon(j,k) \leq \frac{1}{2}\mathcal{C}_i + \epsilon(j,i) - \epsilon(j,k).$$
Similarly,
$$\pi_A(k,j) = \mathcal{C}_k - \epsilon(k,j) \leq \frac{1}{2}\mathcal{C}_i + \epsilon(k,i) - \epsilon(k,j).$$
Using Lemma 3,
$$\begin{aligned}\pi_A(j,k) + \pi_A(k,j) &\leq \left(\frac{1}{2}\mathcal{C}_i + \epsilon(j,i) - \epsilon(j,k)\right) + \\ &\quad \left(\frac{1}{2}\mathcal{C}_i + \epsilon(k,i) - \epsilon(k,j)\right) \\ &\leq \mathcal{C}_i + (\epsilon(j,i) - \epsilon(j,k) + \\ &\quad \epsilon(k,i) - \epsilon(k,j)) \\ &< \mathcal{C}_i + \frac{1}{2}\mathcal{C}_i = 1.5\mathcal{C}_i\end{aligned}$$
Therefore,
$$\begin{aligned}\rho &= \frac{\pi_A(j,k) + \pi_A(k,j)}{\pi_A(i,i) + \pi_A(i,i)} = \frac{\pi_A(j,k) + \pi_A(k,j)}{\mathcal{C}_i} \\ &< \frac{1.5\mathcal{C}_i}{\mathcal{C}_i} = 1.5\end{aligned}$$

Suppose now $\sigma^* = (i,j)$. This time the only hard case is $\sigma^Y = (k,l)$; the four other cases can be treated using the similar argument as before.

Since (i,j) is a PSNE, for any t, $\pi_A(t,i) \leq \pi_A(j,i)$. In particular, $\mathcal{C}_k \leq \pi_A(j,i) + \epsilon(k,i)$.

Hence,

$$\pi_A(k,l) = \mathcal{C}_k - \epsilon(k,l) \leq \pi_A(j,i) + \epsilon(k,i) - \epsilon(k,l).$$

Similarly,

$$\pi_A(l,k) = \mathcal{C}_l - \epsilon(l,k) \leq \pi_A(i,j) + \epsilon(l,i) - \epsilon(l,k).$$

Using Lemma 3,

$$\begin{aligned}\pi_A(k,l) + \pi_A(l,k) &\leq (\pi_A(j,i) + \epsilon(k,i) - \epsilon(k,l)) \\ &\quad + (\pi_A(i,j) + \epsilon(l,i) - \epsilon(l,k)) \\ &\leq (\pi_A(i,j) + \pi_A(j,i)) \\ &\quad + (\epsilon(k,i) - \epsilon(k,l) + \epsilon(l,i) - \epsilon(l,k)) \\ &< (\pi_A(i,j) + \pi_A(j,i)) + \frac{1}{2}\mathcal{C}_i\end{aligned}$$

Similarly,

$$\pi_A(k,l) + \pi_A(l,k) < (\pi_A(i,j) + \pi_A(j,i)) + \frac{1}{2}\mathcal{C}_j.$$

So, it follows that

$$\pi_A(k,l) + \pi_A(l,k) < (\pi_A(i,j) + \pi_A(j,i)) + \frac{1}{4}(\mathcal{C}_i + \mathcal{C}_j).$$

Therefore, by Lemma 1,

$$\begin{aligned}\rho &= \frac{\pi_A(k,l) + \pi_A(l,k)}{\pi_A(i,j) + \pi_A(j,i)} < \frac{(\pi_A(i,j) + \pi_A(j,i)) + (\mathcal{C}_i + \mathcal{C}_j)/4}{\pi_A(i,j) + \pi_A(j,i)} \\ &= \left(\frac{\pi_A(i,j) + \pi_A(j,i)}{\pi_A(i,j) + \pi_A(j,i)}\right) + \frac{1}{2} \cdot \left(\frac{(\mathcal{C}_i + \mathcal{C}_j)/2}{\pi_A(i,j) + \pi_A(j,i)}\right) \\ &\leq 1 + \frac{1}{2} = 1.5\end{aligned}$$

Also, the bound is sharp since 1.5 bound can be achieved using the sequence of graphs shown in Figure 3.

Proof of Theorem 3. Suppose first that (i,i) is a PSNE. Then, the budget multiplier is $1 \leq 2$. So, suppose (i,j) is a PSNE with $i < j$. Then,

$$\pi_A(i,j) \geq \pi_A(j,j) = \mathcal{C}_j/2 \text{ and } \pi_A(j,i) \geq \pi_A(i,i) = \mathcal{C}_i/2.$$

Hence,

$$\mathcal{C}_j/2 \leq \pi_A(i,j) < \mathcal{C}_i \text{ and } \mathcal{C}_i/2 \leq \pi_A(j,i) < \mathcal{C}_j.$$

So,

$$\pi_A(i,j)/\pi_A(j,i) < 2 \text{ and } \pi_A(j,i)/\pi_A(i,j) < 2.$$

Therefore, the budget multiplier is

$$\max\left(\frac{\pi_A(i,j)}{\pi_A(j,i)}, \frac{\pi_A(j,i)}{\pi_A(i,j)}\right) < 2.$$

Also, the bound is sharp since 1.5 bound can be achieved using the sequence of graphs shown in Figure 2. $\square$

Proof of Proposition 2. Suppose (i,j) is a PSNE. Then, $\max(d_i, d_j) = \Delta_1$. Suppose $d_i, d_j < \Delta_1$. Then, there must be $k \neq j$ with $d_k = \Delta_1$ and $\pi_A(k,j) > \pi_A(i,j)$ since

$$\begin{aligned}\pi_A(i,j) &= d_i + (1 - \chi^{-1}_{P(i,j)}) \leq (\Delta_1 - 1) + (1 - \chi^{-1}_{P(i,j)}) &(1)\\ &= \Delta_1 - \chi^{-1}_{P(i,j)} < \Delta_1 \leq d_k + (1 - \chi^{-1}_{P(k,j)}) &(2)\\ &= \pi_A(k,j). &(3)\end{aligned}$$

Also, $\min(d_i, d_j) \geq \Delta_2$. Without loss of generality, suppose $d_i < \Delta_2$ and $d_j = \Delta_1$. Then, using the similar argument to Equation 3, there is $k \neq j$ with $d_k = \Delta_2$ and $\pi_A(k,j) > \pi_A(i,j)$. Therefore, $(i,j) \in \sigma_0 \cup \sigma_1$.

Now, pick an element from σ_0 and σ_1. Suppose that the chosen elements are $(1,1) \in \sigma_0$ and $(2,1) \in \sigma_1$ (with relabeling if necessary); for $(2,1)$, suppose further that $\chi_{P(2,1)} = \max_{(i,j)\in\sigma_1} \chi_{P(i,j)}$ with $d_2 = \Delta_2$ and $d_1 = \Delta_1$.

Lemma 4. *Suppose $i \neq 1$. Then,*

(a) $\pi_A(2,1) \geq \pi_A(i,1)$ (b) and $\pi_B(2,1) \geq \pi_B(2,j)$.

Proof of Lemma 4. For (a), the statement is trivial if $i = 2$. If $i \neq 2$ then $d_i \leq \Delta_3$. If $\Delta_3 = \Delta_2$ then the choice of $(2,1)$ forces the inequality. Otherwise, the inequality follows from the similar argument to Equation (3).

For (b), the statement is trivial if $j = 1$. If $j = 2$ then,

$$\begin{aligned} 2\Delta_1 = \Delta_1 + \Delta_1 \geq \Delta_2 + 1 \quad &\Rightarrow \quad 2\Delta_1 + 2\cdot(1 - \chi^{-1}_{P(2,1)}) \\ &\qquad \geq \Delta_2 + 1 \\ &\Leftrightarrow \quad \pi_B(2,1) = \Delta_1 + 1 - \chi^{-1}_{P(2,1)} \\ &\qquad \geq \frac{\Delta_2 + 1}{2} = \pi_B(2,2) \end{aligned}$$

Otherwise, $d_j \leq \Delta_3$. So, the inequality follows from the similar argument that we used to prove (a). $\square$

By Lemma 4, all PSNE's involve firms seeding at highest and second-highest nodes; that is, it suffices to compare $\pi_A(1,1)$ and $\pi_A(2,1)$ to classify all PSNE's, where $\pi_A(1,1) = \dfrac{\Delta_1 + 1}{2}$ and $\pi_A(2,1) = \Delta_2 + 1 - \chi^{-1}_{P(2,1)}$.

- If $\pi_A(1,1) = \pi_A(2,1)$ then both $(1,1)$ and $(2,1)$ are PSNE's.
- Else if $\pi_A(1,1) > \pi_A(2,1)$ then $(1,1)$ is a PSNE.
- Othewise, $(2,1)$ is a PSNE.

When $\chi_{P(2,1)} = 1$, we need to compare:

$$\frac{\Delta_1 + 1}{2} \text{ and } \Delta_2,$$

or, equivalently,

$$\Delta_1 \text{ and } 2\Delta_2 - 1.$$

If $\Delta_1 > 2\Delta_2 - 1$, $(1,1)$ is a PSNE. If $\Delta_1 < 2\Delta_2 - 1$ then $(2,1)$ is a PSNE. If $\Delta_1 = 2\Delta_2 - 1$ both $(1,1)$ and $(2,1)$ are PSNE's.

Now, when $\chi_{P(2,1)} = 2$, we need to compare:

$$\frac{\Delta_1 + 1}{2} \text{ and } \Delta_2 + \frac{1}{2},$$

or, equivalently,

$$\Delta_1 \text{ and } 2\Delta_2.$$

Finally, when $\chi_{P(2,1)} > 2$, we need to compare:

$$\frac{\Delta_1 + 1}{2} \text{ and } \Delta_2 + 1 - \chi^{-1}_{P(2,1)}$$

or, equivalently,

$$\Delta_1 \text{ and } 2\Delta_2 + \delta$$

where $\delta := 1 - 2\chi^{-1}_{P(2,1)}$. Note that $\delta \in (0,1)$. In particular, Δ_1 cannot be equal to $2\Delta_2 + \delta$ since $\Delta_1, \Delta_2 \in \mathbb{N}$.

Finally, note that $\Delta_1 > 2\Delta_2 + \delta$ if and only if $\Delta_1 \geq 2\Delta_2 + 1$ since $\Delta_1, \Delta_2 \in \mathbb{N}$. Similarly, $\Delta_1 < 2\Delta_2 + \delta$ if and only if $\Delta_1 \leq 2\Delta_2$.

Proof of Proposition 3. Suppose $x \in \sigma_1^*$. Then, with relabeling if necessary, x is $(1,2)$ where $d_1 = \Delta_1$ and $d_2 = \Delta_2$. Now, pick $y = (i,j)$ where $i, j \in V$ and $d_i \geq d_j$. Let $\Delta_x = \Delta_1 + \Delta_2$ and $\Delta_y = d_i + d_j$.

Suppose $Y(y) > Y(x)$. Then, $i \neq j$. If $i = j$ then $Y(y) = d_i + 1$, which forces

$$Y(x) = \Delta_1 + \Delta_2 + 2\cdot(1 - \chi^{-1}_{P(1,2)}) \geq \Delta_1 + \Delta_2 \geq d_i + \Delta_2 \geq d_i + 1 = Y(y).$$

Suppose $Y(y) = Y(x)$. If $i = j$ then

$$\Delta_1 + \Delta_2 + 2\cdot(1 - \chi^{-1}_{P(1,2)}) = Y(x) = Y(y) = d_i + 1$$

Since $d_i + 1 \in \mathbb{N}$, $Y(x)$ is either $\Delta_x + 1$ or Δ_x because $\chi^{-1}_{P(1,2)} \in (0,1)$. Suppose first that $Y(x) = \Delta_x + 1$. Then, $\Delta_1 + \Delta_2 - 1 = d_i \leq \Delta_1$ forces $\Delta_2 \leq 1$. Since $\Delta_2 \geq 1$, $\Delta_2 = 1$. Thus, $d_i = \Delta_1$ and $\Delta_2 = 1$; that is, the network has to be a star. On a star, $\sigma^Y = \sigma_0 \cup \sigma_1^*$. Now, suppose $Y(x) = \Delta_x$. Then, $\Delta_1 + \Delta_2 = d_i \leq \Delta_1$. Hence, $\Delta_2 \leq 0$. But, since $\Delta_2 \geq 1$, this is clearly a contradiction.

Note that $\Delta_x - \Delta_y \geq 0$. First, suppose $\Delta_x - \Delta_y \geq 2$. Then,

$$Y(y) = d_i + d_j + 2\cdot(1 - \chi^{-1}_{P(i,j)}) < d_1 + d_2 + 2\cdot(1 - \chi^{-1}_{P(1,2)}) = Y(x)$$

if and only if

$$2\chi^{-1}_{P(1,2)} - 2\chi^{-1}_{P(i,j)} < \Delta_x - \Delta_y,$$

which holds since

$$2\chi^{-1}_{P(1,2)} - 2\chi^{-1}_{P(i,j)} < 2\chi^{-1}_{P(1,2)} \leq 2 \leq \Delta_x - \Delta_y.$$

Now, suppose $\Delta_x - \Delta_y = 0$. Then, necessarily $d_i = \Delta_1$ and $d_j = \Delta_2$. Hence, $Y(y) \leq Y(x)$ by the choice of x.

Finally, for the case $\Delta_x - \Delta_y = 1$, first suppose $2 \notin N_1$. Then, $\chi_{P(1,2)} \geq 2$, or $\chi^{-1}_{P(1,2)} \leq 1/2$. Hence,

$$2\chi^{-1}_{P(1,2)} - 2\chi^{-1}_{P(i,j)} < 2\chi^{-1}_{P(1,2)} \leq 1 \leq \Delta_x - \Delta_y.$$

Rearranging this inequality yields

$$Y(x) = \Delta_x + 2\cdot(1 - \chi^{-1}_{P(1,2)}) > \Delta_y + 2\cdot(1 - \chi^{-1}_{P(i,j)}) = Y(y).$$

Finally, suppose $2 \in N_1$. Then,

$$\begin{aligned} Y(y) > Y(x) \quad &\Leftrightarrow \quad \Delta_y + 2\cdot(1 - \chi^{-1}_{P(i,j)}) > \Delta_x \\ &\Leftrightarrow \quad 2\chi^{-1}_{P(i,j)} < 2 - (\Delta_x - \Delta_y) = 1 \\ &\Leftrightarrow \quad \chi_{P(i,j)} > 2 \end{aligned}$$

Hence, if $\delta_1 = 1$, $\delta_2 > 2$, and $\sigma_2^* \neq \emptyset$ then $\sigma^Y = \sigma_2^*$. Otherwise, $\sigma^Y = \sigma_1^*$ unless G is a star; for a star, $\sigma^Y = \sigma_0 \cup \sigma_1^*$. $\square$

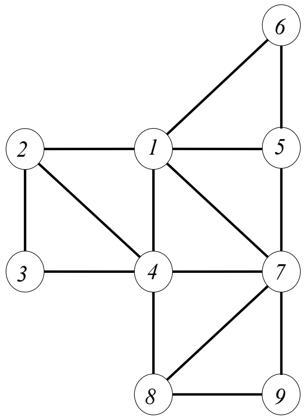
Proof of Corollary 3. It suffices to check explicit characterizations of the equilibria and social optima from Proposition 2 and Proposition 3. $\square$

Proof of Figure 1. $\pi_A(i,j)$ can be computed as:

$$\begin{pmatrix} 2.29 & 3.63 & 4.06 & \mathbf{3.42} & \mathbf{4.03} & \mathbf{4.17} & 3.38 & 3.47 & 3.59 \\ 2.27 & 1.61 & 2.29 & 2.08 & 2.85 & 3.00 & 2.69 & 2.87 & 2.93 \\ 1.99 & 1.59 & 1.24 & 1.53 & 2.21 & 2.33 & 2.12 & 2.28 & 2.32 \\ 3.38 & 3.47 & 3.59 & 2.29 & 3.63 & 4.06 & \mathbf{3.42} & \mathbf{4.03} & \mathbf{4.17} \\ 2.69 & 2.87 & 2.93 & 2.27 & 1.61 & 2.29 & 2.08 & 2.85 & 3.00 \\ 2.12 & 2.28 & 2.32 & 1.99 & 1.59 & 1.24 & 1.53 & 2.21 & 2.33 \\ \mathbf{3.42} & \mathbf{4.03} & \mathbf{4.17} & 3.38 & 3.47 & 3.59 & 2.29 & 3.63 & 4.06 \\ 2.08 & 2.85 & 3.00 & 2.69 & 2.87 & 2.93 & 2.27 & 1.61 & 2.29 \\ 1.53 & 2.21 & 2.33 & 2.12 & 2.28 & 2.32 & 1.99 & 1.59 & 1.24 \end{pmatrix}$$

In particular, we note that $\pi(7,1)$ and $\pi(4,1)$ are different. We show this by an explicit calculation using counting formula. We assume that the seed of firm B is always fixed at 1. Let's perform a DFS (depth-first search) from a seed

Figure 5: Example of a social network that does not admit a PSNE



4 1	→ 2 1/3	→ 3 1/6			
	3 1/2	→ 2 1/6			
	→ 5 1/3	→ 6 1/6	→ 7 1/30	→ 8 1/90	→ 9 1/180
		→ 7 1/15	→ 6 1/30		
			→ 8 1/45	→ 9 1/90	
	→ 7 1/5	→ 5 1/15	→ 6 1/30		
		→ 6 1/10	→ 5 1/30		
		→ 8 1/15	→ 9 1/30		

Table 1: $\pi(4, 1)$

7 1	→ 4 1/5	→ 2 1/15	→ 3 1/30		
		3 1/10	→ 2 1/30		
		5 1/15	→ 6 1/30		
	→ 5 1/5	→ 4 1/15	→ 2 1/45	→ 3 1/90	
			→ 3 1/30	→ 2 1/90	
		→ 6 1/6			
	→ 6 1/2	→ 5 1/6	→ 4 1/30	→ 2 1/90	→ 3 1/180
				3 1/60	→ 2 1/180
	→ 8 1/3	→ 9 1/6			

Table 2: $\pi(7, 1)$

of firm A in a lexicographic order. There are 20 distinct paths in the graph emanating from 4 that avoids 1 as shown in Table 1.

It is immediate from here that:

$$\begin{aligned}\pi(4,1) &= 1 + \frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 2 + \frac{1}{5}\cdot 1 + \frac{1}{6}\cdot 3 + \frac{1}{10}\cdot 1 + \frac{1}{15}\cdot 3 + \frac{1}{30}\cdot 5 \\ &\quad + \frac{1}{45}\cdot 1 + \frac{1}{90}\cdot 2 + \frac{1}{180}\cdot 1 \\ &= \frac{203}{60} \approx 3.383\end{aligned}$$

There are 23 distinct paths in the graph emanating from 7 that avoids 1 as shown in Table 2.

It is immediate from here that:

$$\begin{aligned}\pi(4,1) &= 1 + \frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 2 + \frac{1}{5}\cdot 1 + \frac{1}{6}\cdot 3 + \frac{1}{10}\cdot 1 + \frac{1}{15}\cdot 3 + \frac{1}{30}\cdot 5 \\ &\quad + \frac{1}{45}\cdot 1 + \frac{1}{60}\cdot 1 + \frac{1}{90}\cdot 3 + \frac{1}{180}\cdot 2 \\ &= \frac{41}{12}\left(= \frac{205}{60}\right) \approx 3.417\end{aligned}$$

So, the difference of the payoffs is precisely:

$$\pi(7,1) - \pi(4,1) = \frac{1}{60}\cdot 1 + \frac{1}{90}\cdot 1 + \frac{1}{180}\cdot 1 = \frac{1}{30}$$

Now, it is clear from here that when 7 is the best response for 1, 4 is for 1, and 1 is for 4 by symmetry. Therefore, we conclude that there cannot be a PSNE in Figure 1. □

Proof of Figure 4. Here are all PSNE's:

- $\sigma_1^* = (i, j)$ and $\sigma_{1'}^* = (j, i)$ with $Y(\sigma_1^*) = 5$.
- $\sigma_2^* = (i, k)$; $\sigma_{2'}^* = (k, i)$ with $Y(\sigma_2^*) = 5$.
- $\sigma_3^* = (i, i)$ with $Y(\sigma_3^*) = 4$.

Here is the unique social optimal solution:

- $\sigma_1^Y = (j, k)$ and $\sigma_{1'}^Y = (k, j)$ with $Y(\sigma_1^Y) = 16/3$.

In particular, note that none of the PSNE's is efficient since $Y(\sigma_i^*) < Y(\sigma_j^Y)$ for any valid pair (i, j). □