The Superposition-Traffic Game

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ABSTRACT

We consider a queuing game where a set of sources or players strategically send jobs to a single shared server. The traffic sources have disparate coefficients of variation of the inter-arrival times, and the sources are strategic in their choice of mean inter-arrival times (or the arrival rates). For every job completion, each player receives a benefit that potentially depends on the number of other players using the server (capturing *network effects* due to using the same server). However, the players experience a delay due to their jobs waiting in the queue. Assuming the service times have a general distribution with a finite second moment, we model the delay experienced by the superposed traffic using a Brownian approximation.

In our first contribution, we show that the total rate of job arrivals at a Nash equilibrium with n sources is *larger* when the sources have heterogeneous coefficients of variation, while the average delay experienced by a job is *smaller*, compared to the equilibrium with an equal number of homogeneous sources. In the second contribution, we characterize the equilibrium behavior of the queuing system when the number of homogeneous sources scales to infinity in terms of the rate of growth of the benefits due to network effects.

1. INTRODUCTION

In this paper, we introduce the *Superposition-Traffic Game* to model the strategic behavior of traffic sources in the presence of positive externalities and congestion. In many situations with shared resources, traffic sources derive a positive network effect from the presence of other participants. However, with more participants, congestion is liable to increase as well. For example, at internet exchange points (IXPs) where multiple autonomous systems (ASes) interconnect, more ASes provide multiple alternate routes but can also result in increased congestion at the switches. Similarly, many cloud based services (such as online gaming platforms) exhibit network effects with more participating users and congestion effects due to limited bandwidth and service capacity [1, 2].

A common theme in these examples is that the traffic sources are likely to be strategic about the rate at which they contribute traffic to the network. To capture the congestion effect we model the shared resource as a single server queue. The utility the traffic sources obtain from using the resource

is modeled as a tradeoff between the benefit and the average sojourn time, or delay, of a typical job. The total traffic entering the queue is a *superposition* (or summation) of the traffic from each individual source. We model the expected queuing delay from the steady state distribution of the queue captured by a reflected Brownian Motion. We characterize the steady-state Nash equilibrium traffic rates assuming that the sources play pure strategies.

The variances in the inter-arrival times of the traffic sources have been known to have a significant effect on the average delay [3]. However, these second order variabilities are relatively less explored in game-theoretic settings. Concomitantly, we first characterize the Nash equilibrium traffic profile when the coefficients of variations in inter-arrival times of the traffic sources are heterogeneous. In this case, we prove the apparent result that users with smaller coefficients of variations will contribute a larger portion of the total traffic at an equilibrium. More intriguingly, we also prove that the equilibrium mean waiting time with heterogeneous sources is *lower* than the equilibrium mean waiting time with homogeneous sources, even though the total traffic rate is *greater* with the former than the latter. For a given resource capacity, then, heterogeneous sources can provide a higher throughput with lower delay, raising interesting implications for pricing and protocol design in stochastic systems.

In the case of traffic sources with homogeneous coefficients of variations, we also establish the uniqueness and symmetry of the Nash equilibrium rate profile. Consequently, if the benefit to the source is nondecreasing in the number of sources, then the total equilibrium traffic rate is monotone increasing. It follows that an asymptotically unbounded benefit is a necessary and sufficient condition for the total equilibrium traffic rate to approach the service rate. The scaling of the positive externality has a direct bearing on the scaling of the total equilibrium traffic rate. In particular, we show that with heterogeneous and bounded coefficients of variation, if the benefit grows faster than n (i.e., $\Omega(n)$), then the queue enters heavy-traffic rapidly. On the other hand, if the benefit belongs to $o(\sqrt{n})$, then the queue enters heavy-traffic more slowly, and thus has a much smaller operational time-scale.

1.1 Related Work

The setting considered here is related to some recent papers that incorporate positive externalities in the utilities of users in game-theoretic studies of shared (queuing) systems. Johari and Kumar studied shared platforms, such as online gaming platforms, which exhibit both congestion and network effects [1]. Nair et. al. consider the problem of optimal

capacity provisioning by a profit maximizing firm where the users are strategic and derive greater utilities with increased usage [4, 2]. The setting considered in [4, 2] is a *nonatomic* game, while we consider *atomic* players. Equilibria under heavy traffic was investigated in [5], where users (belonging to different classes) bid for a share of the server. However, the heavy traffic condition was exogenously imposed in that paper, unlike in our work where the equilibrium traffic intensity approaches 1 as a consequence of positive externalities and strategic behavior.

Equilibria with atomic users strategically choosing their arrival rates has been explored in a multitude of queueing models; [6, Chapter 4.6] contains an excellent review. However, the arrival process is often assumed to be a Poisson process. In addition, arrival timing games, where a finite number of users choose a time to arrive at the queue such that their expected waiting time is minimized, are also related to this paper [7, 8]. While each of these models study the effect of multiple users strategically arriving at a queue, the traffic model is not a superposition process. Consequently, the impact of second order variabilities of the traffic on the equilibrium have not been explored in either of the above lines of work.

2. SUPERPOSITION TRAFFIC GAME

Consider a $\sum_j G_j/G/1$ queue where a finite number (say, n) of traffic sources each send traffic to a shared server with a First-In First-Out (FIFO) service discipline. Assume that the sources are statistically independent of each other. The arrival process from source i is a renewal counting process denoted as $A_i(t)$. The traffic from each source has a general inter-arrival time distribution with a finite second moment. Each source controls its traffic rate, λ_i, (or equivalently, the mean inter-arrival time), while the squared coefficient of variation, $c_{a_i}^2$, of the inter-arrival times is fixed. Traffic sources that are naturally bursty, such as packet arrivals from digital voice sources [3], exhibit these characteristics. Service times are generally distributed with a finite mean, μ, and finite squared coefficient of variation, c_s^2. On arrival, if the server is occupied, traffic waits for service in an infinite-sized buffer. The superposition arrival process at the buffer is defined as $A(t) \triangleq \sum_{i=1}^n A_i(t)$.

Since the inter-arrival time at each source i has finite first and second moments, $A_i(t)$ satisfies the Functional Strong Law of Large Numbers (FSLLN) and the Functional Central Limit Theorem (FCLT) [9, Chapter 6]. Since the arrival processes from different sources are independent, we have

$$\frac{A(mt)}{m} = \sum_{i=1}^n \frac{A_i(mt)}{m} \to \sum_{i=1}^n \lambda_i t \text{ u.o.c a.s. as } m \to \infty, \text{ and}$$

$$\frac{A(mt) - mt\sum_{i=1}^n \lambda_i}{\sqrt{m}} = \sum_{i=1}^n \frac{A_i(mt) - m\lambda_i t}{\sqrt{m}}$$

$$\Rightarrow \hat{A}(t) \quad \text{as } m \to \infty,$$

where $\Rightarrow$ indicates weak convergence uniformly over compact intervals (u.o.c.) of $[0, \infty)$, and $\hat{A}(t)$ is a driftless Brownian Motion with variance $\sum_{i=1}^n \lambda_i c_{a_i}^2$. We define $\lambda_T \triangleq \sum_{i=1}^n \lambda_i$ as the total arrival rate from all sources. We will restrict the set of feasible traffic rates for the sources such that $\lambda_T < \mu$. This restriction implies that the $\sum_j G_j/G/1$ queue is recurrent, and consequently the steady-state mean

waiting time can be computed by using a reflected Brownian motion (RBM) approximation to the workload process as ([9, Chapter 6]):

$$S(\{\lambda_j\}) = \frac{\sum_{j=1}^n \alpha_j \lambda_j}{(\mu - \lambda_T)}, \qquad (1)$$

where $\alpha_j \triangleq \frac{c_{a_j}^2 + c_s^2}{2\mu}$. We refer to the quantity α_j as the *Dispersion Index* of source j, as it captures the second order variabilities in both arrival and service times.[1] The strategy of source (player) i is to choose its traffic rate λ_i to maximize its utility (to be defined below). In this paper, we assume that the dispersion index α_i is fixed for each player i. Without loss of generality, assume that the players are ordered such that $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$.

Let λ_{-i} be the vector of traffic rates due to sources other than i. When $\lambda_T < \mu$, we define the utility of player i as

$$u_i(\lambda_i, \lambda_{-i}) = \lambda_i \left(B(n) - \frac{\sum_{j=1}^n \alpha_j \lambda_j}{(\mu - \lambda_T)} \right) \qquad (2)$$

$$\triangleq \lambda_i(B(n) - S(\{\lambda_j\})), \qquad (3)$$

where $S(\{\lambda_j\})$ is the steady-state mean waiting time experienced by a job given by the RBM approximation (1). The quantity $B(n)$ is the average benefit derived by any player per completion of a job when there are n independent traffic sources. We assume that $B(n) < \infty$ when $n < \infty$.

When $\lambda_T \geq \mu$, the queue is unstable, and we define $S(\{\lambda_j\}) = \infty$. Accordingly, each player strictly prefers to send no traffic when $\lambda_T \geq \mu$ or when $u_i(\lambda_i, \lambda_{-i}) < 0$. We thus focus on the region of traffic rates where $u_i(\lambda_i, \lambda_{-i}) \geq 0$. From (2), we obtain

$$B(n) - \frac{\sum_{j=1}^n \alpha_j \lambda_j}{(\mu - \lambda_T)} \leq B(n) - \frac{\alpha_1 \lambda_T}{(\mu - \lambda_T)}.$$

Accordingly, we define $\bar{\lambda}_T := \frac{B(n)\mu}{B(n)+\alpha_1} < \mu$ such that at any strategy profile with $\lambda_T \in (\bar{\lambda}_T, \mu)$, the utility in (2) is negative. Therefore, we define the set of feasible arrival rates of a player i as $\Lambda_i := [0, \bar{\lambda}_T]$.

We refer to this game-theoretic formulation as a *Superposition Traffic Game*. With the exception of Proposition 3, the proofs of all our results are omitted due to space constraints.

3. EQUILIBRIUM CHARACTERIZATION

We first show that utility in (2) is concave in λ_i when $\lambda_T \leq \bar{\lambda}_T$. It suffices to show that the mean waiting time $S(\{\lambda_j\})$ is monotonically increasing and convex in λ_i. Therefore, we compute

$$\frac{\partial S(\{\lambda_i\})}{\partial \lambda_i} = \frac{\alpha_i}{\mu - \lambda_T} + \frac{(\sum_{j=1}^n \alpha_j \lambda_j)}{(\mu - \lambda_T)^2} = \frac{\alpha_i + S(\{\lambda_i\})}{\mu - \lambda_T} > 0, \qquad (4)$$

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[1]Note that our approximation of the mean workload process in (1) is different from similar approximations obtained in [3] for $\sum_j G_j/G/1$ queues under superposition arrivals, albeit the expressions have a similar form. In particular, Sriram and Whitt introduced different notions of indices of dispersion to measure the variability of the traffic, and showed that these metrics describe the mean delay in single server queues under superposition arrivals in a wide range of traffic conditions. We leave an exploration of these alternative notions of indices of dispersion for future work.

$$\frac{\partial^2 S(\{\lambda_i\})}{\partial \lambda_i^2} = \frac{2\alpha_i}{(\mu - \lambda_T)^2} + 2\frac{(\sum_{j=1}^n \alpha_j \lambda_j)}{(\mu - \lambda_T)^3} > 0. \tag{5}$$

As a result, the utility of a player defined in (2) is concave in her arrival rate. Therefore, from [10], we conclude that a pure Nash equilibrium exists in a Superposition-Traffic Game with a finite number of strategic sources.

PROPOSITION 1. *Consider a Superposition-Traffic Game with n players. Then, there exists a pure Nash equilibrium. Furthermore, at any Nash equilibrium strategy profile $\{\lambda_i^*\}$, $B(n) - S(\{\lambda_i^*\}) > 0$ and each $\lambda_i^* > 0$.*

The above proposition implies that at an equilibrium, player i's traffic rate must satisfy the first order condition of optimality for the utility function (2) given by

$$B(n) - S(\{\lambda_j^*\}) - \lambda_i^* \frac{\partial S(\{\lambda_j^*\})}{\partial \lambda_i} = 0$$
$$\implies B(n) = S(\{\lambda_j^*\}) + \lambda_i^* \frac{\alpha_i + S(\{\lambda_j^*\})}{\mu - \lambda_T^*}. \tag{6}$$

We have the following result as a consequence.

PROPOSITION 2. *Consider a Nash equilibrium (NE) strategy profile $\{\lambda_i^*\}$ of a Superposition-Traffic Game. Then, if $\alpha_i < \alpha_j$, we have $\lambda_i^* > \lambda_j^*$ and $\alpha_i \lambda_i^* < \alpha_j \lambda_j^*$.*

The above result has the following interpretation: players with a smaller dispersion index (equivalently, smaller coefficient of variation of the inter-arrival time) send a larger fraction of the total traffic at an equilibrium. Nevertheless, the contribution of that player to the overall delay, given in terms of $\alpha_i \lambda_i^*$, is *smaller* compared to a player with larger dispersion index.

In the next result, we will further explore the implications of this behavior by comparing the equilibria that arise in two instances of a Superposition-Traffic Game, each with n players and mean service rate μ. In the first game, denoted Γ_H, the sources have heterogeneous variances, with parameters $\alpha_1 \le \alpha_2 \le \ldots \le \alpha_n$. In the second game Γ_I, all players have identical coefficients of variation, denoted by $\alpha' = \frac{\sum_{i=1}^n \alpha_i}{n}$.

PROPOSITION 3. *Let $\{\lambda_j^i\}_{j \in \{1,2,\ldots,n\}}$ be any NE strategy profile of Γ_i for $i \in \{I, H\}$. Let the resulting equilibrium waiting time and total arrival rate be denoted as S^i and λ_T^i in the game Γ_i. Then we have $S^I \ge S^H$ and $\lambda_T^I \le \lambda_T^H$.*

PROOF. Consider the first order condition of optimality for a player stated in (6). We add the corresponding equations over all players in Γ_H and obtain

$$nB(n) = nS^H + \frac{\sum_{i=1}^n \alpha_i \lambda_i^H}{\mu - \lambda_T^H} + \frac{S^H \lambda_T^H}{\mu - \lambda_T^H}$$
$$= (n+1)S^H + \frac{S^H \lambda_T^H}{\mu - \lambda_T^H}. \tag{7}$$

Similarly for Γ_I, we obtain

$$nB(n) = (n+1)S^I + \frac{S^I \lambda_T^I}{\mu - \lambda_T^I}. \tag{8}$$

Assume by the way of contradiction that $S^I < S^H$. Since the L.H.S. of both of the above equations are identical, we must have $\lambda_T^I > \lambda_T^H$. From Proposition 2, we know that $\lambda_n^H \le \lambda_j^H$ for every $j \in \{1, 2, \ldots, n\}$. Therefore, $\lambda_n^H \le \frac{\lambda_T^H}{n} < \frac{\lambda_T^I}{n}$.

Let $k \le n$ be the smallest index such that $\lambda_k^H \le \frac{\lambda_T^I}{n}$. Equivalently, for $j < k$, we have $\lambda_j^H > \frac{\lambda_T^I}{n}$. From our hypothesis $S^I < S^H$ and $\lambda_T^I > \lambda_T^H$, we obtain

$$S^I(\mu - \lambda_T^I) < S^H(\mu - \lambda_T^H)$$
$$\implies \alpha' \lambda_T^I < \sum_{j=1}^n \alpha_j \lambda_j^H$$
$$\implies \sum_{j=1}^n \alpha_j \frac{\lambda_T^I}{n} < \sum_{j=1}^n \alpha_j \lambda_j^H$$
$$\implies \alpha_n \left(\frac{\lambda_T^I}{n} - \lambda_n^H \right) < \sum_{j=1}^{n-1} \alpha_j \left(\lambda_j^H - \frac{\lambda_T^I}{n} \right)$$
$$\le \alpha_k \sum_{j=1}^{k-1} \left(\lambda_j^H - \frac{\lambda_T^I}{n} \right) + \alpha_k \sum_{j=k}^{n-1} \left(\lambda_j^H - \frac{\lambda_T^I}{n} \right)$$
$$\implies \left(\frac{\lambda_T^I}{n} - \lambda_n^H \right) \le \sum_{j=1}^{n-1} \left(\lambda_j^H - \frac{\lambda_T^I}{n} \right)$$
$$\implies \lambda_T^I \le \lambda_T^H,$$

which is a contradiction. In the second last step, the inequality holds because $\frac{\lambda_T^I}{n} - \lambda_n^H > 0$ (from our initial hypothesis $\lambda_n^H \le \frac{\lambda_T^H}{n} < \frac{\lambda_T^I}{n}$) and thus $\sum_{j=1}^{n-1} \left(\lambda_j^H - \frac{\lambda_T^I}{n} \right) > 0$. □

Proposition 3 concludes that the equilibrium waiting time with homogeneous sources is *at least* as large as the equilibrium waiting time with heterogeneous sources, while the total equilibrium traffic rate is *at most* that with heterogeneous sources. This conclusion has important implications for protocol design, and control and optimization of congestible services. For instance, Proposition 3 shows that traffic rate will be higher than predicted if the heterogeneity of the autonomous systems (ASes) at internet exchange points (IXPs) is ignored, which could lead to inefficiencies due to under-provisioning of capacity.

In the next section, we will discuss the behavior of the total traffic rate at equilibrium as the number of sources n increases. We will assume that the corresponding dispersion indices lie within an interval $[\alpha_L, \alpha_U]$. Under this condition, we obtain the following bounds on the total equilibrium traffic in terms of the equilibrium traffic that arises in games with players having homogeneous dispersion indices equal to α_L and α_U, respectively.

PROPOSITION 4. *Consider a Superposition-Traffic Game Γ with n potentially heterogeneous sources having $\alpha_L = \alpha_1 \le \alpha_2 \le \ldots \le \alpha_n = \alpha_U$. Let S^* and λ_T^* be the equilibrium delay and total arrival rate at any NE of Γ.*

Let Γ_L and Γ_U be two homogeneous Superposition-Traffic Games, with $\alpha_i = \alpha_L$ for every player i in Γ_L and $\alpha_i = \alpha_U$ for every player i in Γ_U. Let the delay and total arrival rate at any NE of Γ_k be S^k and λ_T^k for $k \in \{U, L\}$.

Then we have $S^U \ge S^* \ge S^L$ and $\lambda_T^U \le \lambda_T^* \le \lambda_T^L$.

The above result will be helpful when we analyze the behavior of equilibrium arrival rates as the number of sources $n \to \infty$, since the equilibria in games with homogeneous players are unique and symmetric (shown below).

PROPOSITION 5. *Consider a Superposition-Traffic Game with n homogeneous players, i.e., $\alpha_i = \alpha$ for every player*

i. Then, for any $n < \infty$, the NE strategy profile is unique and symmetric. Furthermore, if $B(n)$ is nondecreasing in n, then $\lambda_{T,n}^*$ is monotonically increasing in n.

4. HEAVY TRAFFIC BEHAVIOR UNDER POSITIVE EXTERNALITIES

In this section, we study how network effects (captured by $B(n)$) influence the behavior of the total equilibrium traffic rate as the number of sources scale to ∞. In particular, $B(n) \to \infty$ as $n \to \infty$ is a necessary and sufficient condition for the total equilibrium traffic rate rate to converge to μ, i.e., the system approaches a heavy traffic regime as the number of sources increase.

PROPOSITION 6. *Consider a Superposition-Traffic Game with potentially heterogeneous sources. Let $\alpha_i \in [\alpha_L, \alpha_U]$ for every player i. Let $\lambda_{T,n}^*$ be the total traffic rate at an equilibrium of the game with n players. Then $\lambda_{T,n}^* \to \mu$ as $n \to \infty$ if and only if $B(n) \to \infty$.*

In the above proposition, we assumed that as the number of players increases, their dispersion indices remain within the bounded interval $[\alpha_L, \alpha_U]$. The proof relies on the interval being independent of the number of users. If, for instance, α_U increases with n, then $\lambda_{T,n}^U$ may exhibit different behavior instead of converging to μ.

When $\lambda_{T,n}^* \to \mu$ and the system approaches a heavy-traffic limit, the behavior of the queuing process depends on the relative speed at which $\lambda_{T,n}^*$ increases to μ and the growth of n. For example, Whitt [11] shows that the queue length process under superposition arrivals (from i.i.d. sources, without any strategic considerations) and heavy traffic limits exhibits very different behavior depending on whether the quantity $n(\mu - \lambda_{T,n}^*)^2$ converges to 0 or ∞. More generally, $n(\mu - \lambda_{T,n}^*)^2$ is often used to scale time while obtaining so-called heavy-traffic approximations. Therefore, we investigate how the rate of growth of the positive externality parameter $B(n)$ affects the convergence of $n(\mu - \lambda_{T,n}^*)^2$.

PROPOSITION 7. *Consider a Superposition-Traffic Game with potentially heterogeneous sources. Let $\alpha_i \in [\alpha_L, \alpha_U]$ for every player i. Let $B(n) \to \infty$ as $n \to \infty$. As $n \to \infty$,*

1. if $n \in o(B(n))$, then $n(\mu - \lambda_{T,n}^*)^2 \to 0$;

2. otherwise, if $B(n) \in o(\sqrt{n})$, $n(\mu - \lambda_{T,n}^*)^2 \to \infty$.

According to Proposition 7, when $B(n)$ grows faster than n, the convergence of $(\mu - \lambda_{T,n}^*)^2$ to 0 is relatively quicker than the increase in n. By Theorem 3 of Whitt [11], the limiting queue length process approaches a $G/G/1$ queue in heavy-traffic. On the other hand, when $B(n)$ grows slower than $\sqrt{n}$, Theorem 4 of Whitt [11] implies that the queue length process approaches the queue length of a $M/G/1$ queue. Note that although the setting in Whitt [11] considers the traffic from different sources to be *i.i.d*, we conjecture that the result also holds when the sources are independent, but not identically distributed, as long as the individual traffic rates scale to 0 as $n \to \infty$.

5. CONCLUSION

We studied a game-theoretic setting where a set of sources send traffic to a shared server by strategically choosing their traffic rates. We established the existence of a pure Nash equilibrium in games with a fixed (finite) number of sources, including when the traffic from different sources have heterogeneous variances in their respective inter-arrival times. The equilibrium traffic rates under heterogeneous sources leads to a *higher* total traffic and *smaller* delay, compared to the equilibrium rates under homogeneous sources. This has implications for routing, capacity provisioning and pricing of stochastic systems. In the second part of the paper, we explored how the nature of positive externalities affects the queue length behavior in the limit as the number of sources (and the benefits due to network effects) scale to infinity. In future, we will study the Superposition-Traffic Game under transient conditions and when the game is played over a network of queues.

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